Quantum mechanics II, Problems 9 : Group averaging and Conjugacy classes

Solutions

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Problem 1 : Conjugacy classes and number of irreducible representations

For the groups C3v and Z_N compute :

1. Their conjugacy classes.

   (a) C3v : From our understanding of the C3v group we could intuitively guess the conjugacy class. We have first the class that contain the identity and only the identity (as for every groups) $\{e\}$. Then we have the two rotations $\{r_1, r_2 = r_1^2\}$ that must form a conjugacy class and the three axial symmetries $\{s_1,\ s_2,\ s_2\}$. Now we have to show it. For e we have $geg^{-1} = e$ as is the case for all groups. Now, note that the composition law for our group can be summarized in the following formulas :

   $$r_i r_j = r_{i+j}, \quad r_i s_j = s_{i+j}, \quad s_i r_j = s_{i-j}, \quad s_i s_j = r_{i-j} \tag{1}$$

   Then for r_1 and r_2 we can show that $s_i r_j s_i = r_{-j}$. Finally, note that $s_i s_j s_i = s_i$ and $r_i s_j r_i^{-1} = s_{2i+j}$. We then have three conjugacy class : $\{e\}$. $\{r_1, r_2\}$ and $\{s_1, s_2, s_3\}$. The proof we provided in here follows for the case of the general Dihedral group, which is the symmetry group of a general regular polygon (you can read more about it in the Wikipedia page). Check it !

   (b) Z_N : The group Z_N is abelian which means that every element commutes with every other element. If we use this we have

   $$uau^{-1} = (ua)u^{-1} = (au)u^{-1} = a(uu^{-1}) = ae = a \tag{2}$$

   This means that the only element in the conjugacy class of a, is a. We then have N conjugacy class $\{i\}\ \forall i \in Z_N$.

2. The number of (non-equivalent) irreducible representations.
   Recalling that the number of irreducible representations is equal to the number of conjugacy classes, $N_r = N_c$ then we have :

   (a) C3V : $N_r = N_c = 3$

   (b) Z_N : $N_r = N_c = N$

3. The possible dimensions of these irreducible representations.
   To find the possible dimensions of the irreducible representations we can use Burnside's lemma :

   $$\sum_{a=1}^{N_r} l_a^2 = |G| \tag{3}$$

   where $|G|$ is the order of the group, $N_r = N_c$ is the number of irreducible representations and for $a = 1, ..., N_r$, l_a is the dimension of the representation a.

(a) C3v : Here we have

$$l_1^2 + l_2^2 + l_3^2 = 6 \tag{4}$$

So the only possible dimensions are 1,1 and, 2.

(b) Z_N : Here we have

$$\sum_{a=1}^{N} l_a^2 = N \tag{5}$$

So the only possible dimensions are $l_a = 1 \; \forall a$.

Problem 2 : Group representation theory applied to dephasing

You already did the first two questions in the last exercise sessions but the answers are useful for the next questions.

1. Prove that the Pauli matrices and the identity (times ± 1, $\pm i$) form a (non-Abelian) group with the matrix product.
   A group has to have different properties
   - Closeness : As we know $\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k$, so the product of two Pauli matrices is a Pauli matrix with a pre-factor of either ± 1, $\pm i$, so the product of each of two possible matrices is in the set of our matrices.
   - Associative : the matrix product is associative.
   - Identity. the group includes the identity matrix.
   - Inverse : $\sigma_i \sigma_i = \mathbb{I}$, $i\sigma_i \times -i\sigma_i = \mathbb{I}$, $-\mathbb{I} \times -\mathbb{I} = \mathbb{I}$. So the inverse of each matrix is in the set of our matrices as well.
2. Prove that if $R(g)$ is a representation of a group G then $R(g) \otimes R(g)$ is also a representation of G.
   Note that

$$R(g_1) \otimes R(g_1) \cdot R(g_2) \otimes R(g_2) = (R(g_1) \cdot R(g_2)) \otimes (R(g_1) \cdot R(g_2)) = R(g_1 g_2) \otimes R(g_1 g_2) \tag{6}$$

3. Consider a unitary irreducible representation $R(g) = U_g$ of group G. Use the Grand Orthogonality Theorem to prove that

$$\frac{1}{N} \sum_g U_g X U_g^\dagger = \frac{1}{d} \mathrm{Tr}[X]\, I \tag{7}$$

   where $d = \dim(X)$ and N is the order of the group.
   Because this representation is irreducible, we can use the Grand Orthogonality theorem and rewrite $\frac{1}{N}\sum_g U_g X U_g^\dagger$ as follows.

$$\frac{1}{N} \sum_g U_g X U_g^\dagger = \frac{1}{N} \sum_{jklm} \sum_g [U_g]_{lm} X_{mj} [U_g^\dagger]_{jk} |l\rangle\langle k| \tag{8}$$

$$= \frac{1}{d} \sum_{jklm} \delta_{lk}\delta_{jm} X_{mj} |l\rangle\langle k| \tag{9}$$

$$= \frac{1}{d} \sum_{jk} X_{jj} |k\rangle\langle k| \tag{10}$$

$$= \frac{1}{d} \mathrm{Tr}[X] I \tag{11}$$

4. Use this result to (carefully!) explain why randomly applying either I (i.e, do nothing), σ_x, σ_y, or σ_z (with equal probability) to any single qubit state on average results in the maximally mixed state.
We can consider the group of Pauli matrices and identity with ± 1 and $\pm i$ prefactors that we had in the first part of the question and use the result in the third part to write the average of X.

$$\frac{1}{N}\sum_g U_g \rho U_g^\dagger = \frac{1}{N}(4I\rho I + 4\sigma_x\rho\sigma_x + 4\sigma_y\rho\sigma_y + 4\sigma_z\rho\sigma_z) \tag{12}$$

$$= \frac{1}{4}(I\rho I + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z) \tag{13}$$

where $N = 16$ is the order of the group. So averaging over all elements of the group is equal to randomly applying either I, σ_x, σ_y, or σ_z with probability $\frac{1}{4}$ to any single qubit state. And then from the previous part, we know that it is equal to the maximally mixed state.

$$\frac{1}{4}(I\rho I + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z) = \frac{1}{2}\mathrm{Tr}[\rho]I = \frac{1}{2}I \tag{14}$$

5. Consider now instead a completely reducible unitary representation $U_g = \oplus_k R_k(g)$ where the $R_k(g)$ are d_k dimensional unitary irreducible representations. It can be shown that

$$\langle X\rangle_G = \frac{1}{N}\sum_g U_g X U_g^\dagger = \sum_k \frac{\mathrm{Tr}[X\Pi_k]}{d_k}\Pi_k. \tag{15}$$

What are Π_k and d_k in this expression?
See Chapter 9 - p. 124-125 (Eq.(9.71)). We can show that

$$\langle X\rangle_G = \frac{1}{N}\sum_g U_g X U_g^\dagger = \sum_k \frac{\mathrm{Tr}[X\Pi_k]}{d_k}\Pi_k = \bigoplus_k \frac{\mathrm{Tr}[X\Pi_k]}{d_k} I_k\,. \tag{16}$$

Π_x are the projectors to the subspace the irreducible representations is acting on and d_x are their dimensions. $\Pi_x = \sum_i^{d_x} |x,i\rangle\langle x,i|$. In other words, d_x is the rank of Π_x, and Π_x is a projector into the subspace of the irrep x.

6. The above relation for averaging over representations of finite groups, Eq. (15), generalizes to averaging over compact Lie groups. In this case the finite average $\frac{1}{N}\sum_g$ becomes a continuous integral over a uniform measure $\int d\mu(g)$ and we have :

$$\langle X\rangle_G := \int_G d\mu(g) U_g X U_g^\dagger = \bigoplus_k \frac{\mathrm{Tr}[X\Pi_k]}{d_k} I_k. \tag{17}$$

Use this result to derive an explicit expression (i.e. compute the relevant d_k and Π_k) for the averaged state ρ that results from randomly evolving ρ under the tensor product of two random single qubit unitaries. That is, from apply $U\otimes U$ with $U \in \mathrm{U}(2)$, to any two qubit state ρ, and then averaging :

$$\langle\rho\rangle = \int_{U(2)} d\mu\, U\otimes U\,\rho\, U^\dagger \otimes U^\dagger\ . \tag{18}$$

The easiest way to do this is to find something that commutes with this and that we know how to diagonalise. Then we can use that basis. As we have seen in class $[U \otimes U, \text{SWAP}] = 0$, therefore we can use the SWAP basis to find the projectors of the irreductible representation. The basis of the SWAP is the symmetric and antisymmetric spaces, respectively with eigenvalue $\lambda_+ = 1$

$$|\phi_0\rangle = |00\rangle \tag{19}$$
$$|\phi_1\rangle = |11\rangle \tag{20}$$
$$|\phi_2\rangle = \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) \tag{21}$$
$$\tag{22}$$

and with eigenvalue $\lambda_- = -1$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) \tag{23}$$

the corresponding dimensions are $d_+ = 3$, $d_- = 1$. Therefore, from 5. we have

$$\langle\rho\rangle = \frac{1}{3}\text{Tr}\left[\sum_{i=0}^{2} |\phi_i\rangle\langle\phi_i| \, \rho\right] I_3 \oplus \text{Tr}[|\phi_3\rangle\langle\phi_3| \, \rho] I_1 \tag{24}$$

7. Hence (or otherwise) compute the states that result from averaging (i.e, compute $\langle\rho\rangle$ in Eq. (18)) for the following states :
   i. $\rho = |\Phi^+\rangle\langle\Phi^+|$ with $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
   ii. $\rho = |\Psi^-\rangle\langle\Psi^-|$ with $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$
   iii. $\rho = |00\rangle\langle 00|$
   iv. An arbitrary tensor product two qubit state $\rho \otimes \sigma$ (hint : use the Bloch vector representation).
   i. This state is in the symmetric space, thus

$$\langle\rho\rangle = \frac{1}{3} I_3 \oplus 0$$

ii. This state is in the anti-symmetric space

$$\langle\rho\rangle = 0_3 \oplus 1$$

iii. This state is in the symmetric space, thus

$$\langle\rho\rangle = \frac{1}{3} I_3 \oplus 0$$

iv. There are several ways of solving this problem, but some of the are very cumbersome. Let's try and be smart. We start by denoting $\rho = \frac{1}{2}(I + n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)$, $\sigma = \frac{1}{2}(I + m_x\sigma_x + m_y\sigma_y + m_z\sigma_z)$. Now, by applying the formula from the previous sections that links the average to the trace of the invariant subspaces (irreducible components), we find that we only need to compute the traces. Moreover, note that it is actually very easy to calculate the total trace as :

$$\text{Tr}(\rho \otimes \sigma) = \frac{\text{Tr}(I \otimes I)}{4} + \frac{\text{Tr}(I \otimes \mathbf{m}.\sigma)}{4} + \frac{\text{Tr}(\mathbf{n}.\sigma \otimes I)}{4} + \frac{\text{Tr}(\mathbf{n}.\sigma \otimes \mathbf{m}.\sigma)}{4} = 1+0+0+0 = 1 \tag{25}$$

Now let us find the projector onto the anti-symmetric space (Note that it is also correct to try and do this for the symmetric space but it takes longer) :

$$\Pi_{\text{anti-sym}} = |\phi_3\rangle \langle\phi_3| \tag{26}$$

Now, let us calculate the final piece :

$$\begin{aligned}\text{Tr}(\rho \otimes \sigma\Pi_{\text{anti-sym}}) = &\frac{\text{Tr}(I \otimes I\Pi_{\text{anti-sym}})}{4} + \frac{\text{Tr}(I \otimes \mathbf{m}.\sigma\Pi_{\text{anti-sym}})}{4} \\ &+ \frac{\text{Tr}(\mathbf{n}.\sigma \otimes I\Pi_{\text{anti-sym}})}{4} + \frac{\text{Tr}(\mathbf{n}.\sigma \otimes \mathbf{m}.\sigma\Pi_{\text{anti-sym}})}{4}\end{aligned} \tag{27}$$

The first term is trivially equal to $\frac{1}{4}$. You can check (convince yourself!) that the other terms have zero contribution unless for the case where $\sigma_x \otimes \sigma_x$, $\sigma_y \otimes \sigma_y$ and, $\sigma_z \otimes \sigma_z$. Therefore, we have :

$$\text{Tr}(\rho \otimes \sigma\Pi_{\text{anti-sym}}) = \frac{1 - \mathbf{n.m}}{4} \tag{28}$$

As the sum of the two traces must be equal to the total trace 1, we also have :

$$\text{Tr}(\rho \otimes \sigma\Pi_{\text{sym}}) = \frac{3 + \mathbf{n.m}}{4} \tag{29}$$

Then the average state will be

$$\langle\rho \otimes \sigma\rangle = \frac{1}{3}\left(\frac{3 + \mathbf{n.m}}{4}\right) I_3 \oplus \left(\frac{1 - \mathbf{n.m}}{4}\right) I_1$$

Alternatively for 6. and 7., we can use properties of the SWAP operator as follows. First, in **6.** notice that the projectors onto the symmetric and antisymmetric subspaces are respectively given by

$$\Pi_{\text{sym}} := \sum_{i=0}^{2} |\phi_i\rangle\langle\phi_i| = \frac{\mathbb{1} \otimes \mathbb{1} + \text{SWAP}}{2} \tag{30}$$

$$\Pi_{\text{anti-sym}} := |\phi_3\rangle\langle\phi_3| = \frac{\mathbb{1} \otimes \mathbb{1} - \text{SWAP}}{2} \ . \tag{31}$$

Therefore, we have

$$\begin{aligned}\langle\rho\rangle &= \frac{1}{3}\text{Tr}\left[\Pi_{\text{sym}}\rho\right]\Pi_{\text{sym}} + \text{Tr}\left[\Pi_{\text{anti-sym}}\rho\right]\Pi_{\text{anti-sym}} &(32)\\ &= \frac{\mathbb{1} \otimes \mathbb{1}}{6}\left(2\text{Tr}[\rho] - \text{Tr}[\text{SWAP}\rho]\right) + \frac{\text{SWAP}}{6}\left(2\text{Tr}[\text{SWAP}\rho] - \text{Tr}[\rho]\right) &(33)\\ &= \frac{\mathbb{1} \otimes \mathbb{1}}{6}\left(2 - \text{Tr}[\text{SWAP}\rho]\right) + \frac{\text{SWAP}}{6}\left(2\text{Tr}[\text{SWAP}\rho] - 1\right) \ , &(34)\end{aligned}$$

where the last equality holds since ρ is a quantum state.
From this expression, averaging over the states in **7.** is quite straightforward. Indeed, as states ρ in both **i.** and **iii.** are in the symmetric subspace we have $\text{Tr}[\rho\text{SWAP}] = 1$ which leads to

$$\langle\rho\rangle = \frac{\mathbb{1} \otimes \mathbb{1} + \text{SWAP}}{6} = \frac{1}{3}\Pi_{\text{sym}} \ . \tag{35}$$

Similarly, the state in **ii.** is in the antisymmetric subspace. In this case, we have $\text{Tr}[\rho\text{SWAP}] = -1$ which gives

$$\langle\rho\rangle = \frac{\mathbb{1} \otimes \mathbb{1} - \text{SWAP}}{2} = \Pi_{\text{anti-sym}} = \rho \ . \tag{36}$$

Finally, for **iv**. we use the property $\mathrm{Tr}[(A\otimes B)\mathrm{SWAP}] = \mathrm{Tr}[AB]$ for any operators A and B (you can verify it your-self as an exercise). This leads to

$$\langle \rho\otimes\sigma\rangle = \frac{\mathbb{1}\otimes\mathbb{1}}{6}\left(2-\mathrm{Tr}[\rho\sigma]\right) + \frac{\mathrm{SWAP}}{6}\left(2\mathrm{Tr}[\rho\sigma]-1\right)\ . \tag{37}$$

Moreover, if we define the Bloch vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ corresponding respectively to states ρ and σ, we can easily show (exercise !) that

$$\mathrm{Tr}[\sigma\rho] = \frac{1+\mathbf{r}_1\cdot\mathbf{r}_2}{2}\ . \tag{38}$$

This leads to

$$\langle \rho\otimes\sigma\rangle = \frac{\mathbb{1}\otimes\mathbb{1}}{12}\left(3-\mathbf{r}_1\cdot\mathbf{r}_2\right) + \frac{\mathrm{SWAP}}{6}\mathbf{r}_1\cdot\mathbf{r}_2\ . \tag{39}$$

Can you relate this with the previous answer ?

More useful properties of the SWAP operator

You can further rewrite it in the Pauli basis (more conventional) using the Pauli decomposition of SWAP i.e.

$$\mathrm{SWAP} = \frac{1}{2}\sum_{\sigma\in\{\mathbf{1},\sigma_x,\sigma_y,\sigma_z\}} \sigma\otimes\sigma\ , \tag{40}$$

which you can try to prove as an exercise. More generally, if SWAP acts on two n-qubits space (i.e. SWAP is of dimension 2^{2n}), we have

$$\mathrm{SWAP} = \frac{1}{2^n}\sum_{\sigma\in\{\mathbf{1},\sigma_x,\sigma_y,\sigma_z\}^{\otimes n}} \sigma\otimes\sigma\ . \tag{41}$$

(hint : use $\mathrm{Tr}[(A\otimes B)\mathrm{SWAP}] = \mathrm{Tr}[AB]$ with $A\otimes B\in\{\mathbb{1},\sigma_x,\sigma_y,\sigma_z\}^{\otimes 2n}$.)